Common Solution of the Energy Dissipation Problem in Shafts Under Time-Dependent Angles of Twist

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Abstract

The present paper deals with the energy dissipation problem in viscoelastic continuously inhomogeneous stepped shafts under time-dependent angles of twist. The shafts analyzed in the paper have circular cross-section. Common solution of the energy dissipation problem is derived. Statically determinate as well as statically indeterminate shafts are considered. The viscoelastic behavior of the shafts is treated by models representing systems of springs and dashpots under time-dependent shear strain. The shafts are continuously inhomogeneous along the radius of the cross-section. Because of this, the shaft properties are continuously distributed along the radius. The common solution for the energy dissipation is obtained by analyzing the stresses and strains in the dashpots of the viscoelastic models (actually, this approach uses the fact that in models with springs and dashpots the energy is dissipated by the dashpots). An example illustrating the application of the common solution is presented. The dissipated energy (DE) is derived also by direct integration in the time domain for verification. The DE in the statically determinate shafts is compared with this in an indeterminate shaft. It is demonstrated that the common solution is applicable also when the shafts are under angles of twist whose number is less that the number of the shaft portions.

Keywords: *Energy dissipation, Common solution, Stepped shaft, Angle of twist, Dashpot, Material inhomogeneity*

1. Introduction

Structural members of circular cross-section (shafts, axles) subjected to torsion are applied as components of various machines, mechanisms and constructions in different areas of engineering [1 - 3]. Development of analyses which consider different aspects of the viscoelastic behavior of the shafts is necessary for guaranteeing the fulfillment of their load resisting purposes.

One of the specific peculiarities of the behavior of shafts made of viscoelastic engineering materials is the energy dissipation. When the shafts mechanical behavior is treated by viscoelastic models representing systems of springs and dashpots, the energy is dissipated by the dashpots.

The aim of the present paper is to derive common solution of the energy dissipation problem in stepped shafts which are subjected to angles of twist varying continuously with time. It should be noted that for the time being the energy dissipation analyses of shafts under torsion deal with separate cases and no attempt for deriving common solution of this problem has been made [4 - 7]. Thus, the present paper fills up this emptiness in the scientific investigations. The shafts studied here have circular cross-section. Besides, since using of continuously inhomogeneous (functionally graded) materials for manufacturing of various structural components is an important trend in the modern engineering [8 - 10], it is assumed that the shafts studied here are continuously inhomogeneous along the radius of the cross-section. Shafts of common configuration with arbitrary number of portions of different radius of cross-section are considered. The shafts viscoelasic behavior is treated by models having arbitrary number of springs and dashpots. Common solution of the energy dissipation is derived for both statically determinate and statically indeterminate shafts under an arbitrary set of varying with time angles of twist. The common solution is applied for a shaft with two portions under angles of twist varying in sine law with time. The solution is verified by using an approach for determination of the DE based on a direct integration in the time domain. Results indicating how the energy dissipation is influenced by the shaft size, the loading parameters and the material inhomogeneity are presented.

2. Common Solution

The stepped shaft to be considered first in this section of the paper is given in Figure 1.

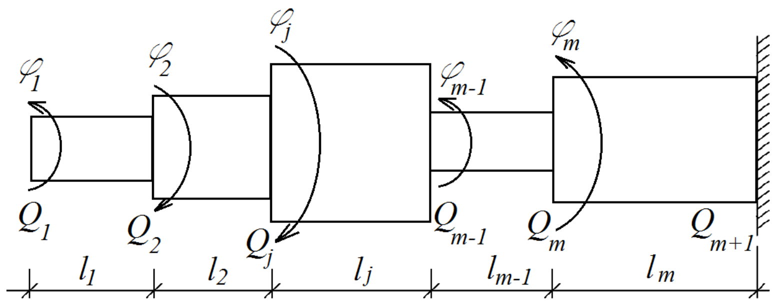

Fig. 1. Common schema of stepped shaft.

For generality the number of portions in the shaft is taken arbitrary (the number is denoted by m). The shaft has circular cross-section. The radius of the cross-section of the *j*-th portion of the shaft is R_j. The length of the *j*-th portion is l_j. The shaft right end is rigidly fixed. The material of the shaft is continuously inhomogeneous along R_j. The shaft is under angles of twist, $\varphi_1, \varphi_2, ..., \varphi_j, ..., \varphi_m$, as shown in Figure 1. The laws for change of the angles of twist with time, t, are known in advance. The angles of twist are related to the shear strains, γ_{pfj}, on the shaft surface by applying the integrals of Maxwell-Mohr. Thus, φ_j is

$$\varphi_j = \sum_{i=j}^{i=m} \frac{\gamma_{pfj}}{R_j} l_j, \tag{1}$$

where

$$j = 1, 2, \ldots, m. \tag{2}$$

Relations (1) can be used to determine γ_{pfj} for a given set of φ_j. The shear strains, γ_{shj}, in the *j*-th portion of the shaft can be presented by

$$\gamma_{shj} = \frac{\gamma_{pfj}}{R_j} R, \tag{3}$$

where

$$0 \le R \le R_j. \tag{4}$$

Determination of γ_{pfj} and γ_{shj} where $j = 1, 2, \ldots, m$ is a preliminary work that should be done before deriving the DE in the shaft under consideration.

The viscoelastic behavior of the shaft is treated by a viscoelastic model representing a system of springs and dashpots. In such models, the energy is dissipated by the dashpots. The number of dashpots in the model is n. Thus, the specific strain energy, u_{0j}, in the dashpots for the *j*-th portion of the shaft is given by

$$u_{0j} = \sum_{i=1}^{i=n} \frac{1}{2} \tau_{\eta ji} \gamma_{shji}, \tag{5}$$

where γ_{shji} is found by (3), $\tau_{\eta ji}$ is the shear stress in the *i*-th dashpot. The subscript, j, in (5) refer to the *j*-th portion of the shaft. $\tau_{\eta ji}$ is defined by

$$\tau_{\eta ji} = \eta_{ji} \dot{\gamma}_{shji}, \tag{6}$$

where η_{ji} is the coefficient of viscosity, $\dot{\gamma}_{shji}$ is the first derivative with respect to time.

The DE, U, in the shaft is found by integrating u_{0j} in the shaft portions

$$U = \sum_{j=1}^{j=m} l_j \iint_{(A_j)} u_{0j} dA, \tag{7}$$

where A_j is the area of the cross-section of the *j*-th shaft portion.

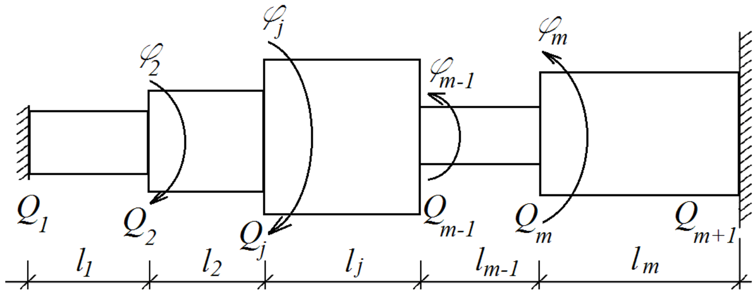

Fig. 2. Common schema of statically indeterminate stepped shaft.

Common solution of the DE is obtained also for the statically indeterminate shaft in Figure 2. The shear strains, γ_{pfj}, on the shaft surface can be derived from the following equations:

$$\varphi_1 = \sum_{j=1}^{j=m} \frac{\gamma_{pfj}}{R_j} l_j = 0, \tag{8}$$

$$\varphi_j = \sum_{j=2}^{j=m} \frac{\gamma_{pfj}}{R_j} l_j \ , \tag{9}$$

where φ_j is a given set of angles of twist. Equation (8) reflects the fact that the angle of twist, φ_1, of the shat left end is zero due to the clamping. Then the DE can be determined by applying (5), (6) and (7).

3. Illustrative Example

First, a statically determinate shaft is considered (Figure 3).

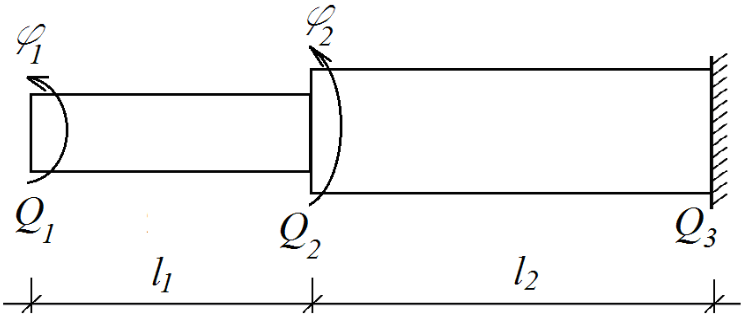

FIGURE 3. Stepped shaft with two portions.

The shaft has two portions, Q_1Q_2 and Q_2Q_3. The shaft is under torsion so that the angles of twist, φ_1 and φ_2, vary with time according the following dependences:

$$\varphi_1 = \varphi_{1B} \sin(\omega t), \tag{10}$$

$$\varphi_2 = \varphi_{2B} \sin(\omega t), \tag{11}$$

where φ_{1B}, φ_{2B} and ω are parameters.

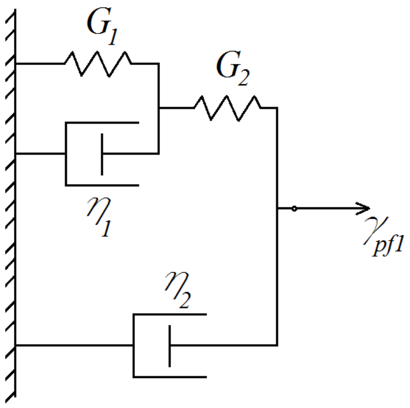

Fig. 4. Schema of viscoelastic model.

By using (1), (10) and (11), we derive

$$\gamma_{pf1} = \frac{R_1}{l_1}(\varphi_2 - \varphi_1),\ \gamma_{pf2} = \frac{\varphi_2 R_2}{l_2} \ . \tag{12}$$

The specific strain energy in portion, Q_1Q_2, of the shaft is determined first. The viscoelastic behavior of the shaft is described by the model shown in Figure 4. The model is a system of two springs and two dashpots. The model is under shear strain, γ_{pf1}, that is expressed by the first formula in (12). By analyzing the equilibrium of the springs and dashpots in the viscoelastic model in Figure 4 and after performing mathematical manipulations, we obtain

$$\gamma_{sh11} = \frac{R}{R_1}\left[\ -\delta_5 e^{-\delta_2 t} + \delta_4 \sin(\omega t) + \delta_5 \cos(\omega t)\ \right], \tag{13}$$

$$\tau_{\eta 11} = \frac{\eta_1 R}{R_1}\left[\delta_2 \delta_5 e^{-\delta_2 t} + \delta_4 \omega \cos(\omega t) - \delta_5 \omega \sin(\omega t)\right], \tag{14}$$

$$\gamma_{sh12} = \frac{\gamma_{pf1}}{R_1} R, \tag{15}$$

$$\tau_{\eta 12} = \frac{\eta_2 \delta_1 \omega R \cos(\omega t)}{R_1}, \tag{16}$$

where

$$\delta_1 = \frac{R_1}{l_1}\left(\varphi_{1B} - \varphi_{2B}\right),\ \delta_2 = \frac{G_1 + G_2}{\eta_1},\ \delta_3 = \frac{\delta_1 G_2}{\eta_1}, \tag{17}$$

$$\delta_4 = \frac{\delta_2 \delta_3}{\omega^2 + \delta_2^2},\ \delta_5 = -\frac{\delta_3 \omega}{\omega^2 + \delta_2^2}. \tag{18}$$

In formulas (13), (14), (15) and (16), γ_{sh11} and $\tau_{\eta 11}$ are the strain and the stress in the dashpot with coefficient of viscosity, η_1, while γ_{sh12} and $\tau_{\eta 12}$ are the strain and the stress in the dashpot with coefficient of viscosity, η_2.

In formulas (13), (14), (15), (16), (17) and (18), G_1 and G_2 are the shear moduli of the springs, η_1 and η_2 are the coefficients of viscosity of the dashpots (Figure 4). The distributions of G_1, G_2, η_1 and η_2 along the radius of the shaft cross-section are

$$G_1 = G_{1pf} + \frac{G_{1cr} - G_{1pf}}{R_1^{\mu_1}}\left(R_1 - R\right)^{\mu_1},\ G_2 = G_{2pf} + \frac{G_{2cr} - G_{2pf}}{R_1^{\mu_2}}\left(R_1 - R\right)^{\mu_2}, \tag{19}$$

$$\eta_1 = \eta_{1pf} + \frac{\eta_{1cr} - \eta_{1pf}}{R_1^{\mu_3}}\left(R_1 - R\right)^{\mu_3},\ \eta_2 = \eta_{2pf} + \frac{\eta_{2cr} - \eta_{2pf}}{R_1^{\mu_4}}\left(R_1 - R\right)^{\mu_4}, \tag{20}$$

where subscripts, pf and cr, refer to the surface and the centre of the shaft, respectively, μ_1, μ_2, μ_3 and μ_4 are parameters.

The specific strain energy in the two dashpots for shaft portion, Q_1Q_2, is found by using (5), i.e.

$$u_{01} = \frac{1}{2}\tau_{\eta 11}\gamma_{sh11} + \frac{1}{2}\tau_{\eta 12}\gamma_{sh12}. \tag{21}$$

In formula (21), γ_{sh11}, $\tau_{\eta 11}$, γ_{sh12} and $\tau_{\eta 12}$ are determined by (13), (14) and (15). Formula (21) is applied also to obtain the specific strain energy, u_{02}, in the two dashpots for the shaft portion, Q_2Q_3. For this purpose, γ_{pf1}, l_1, R_1, φ_{1B} and φ_{2B} are replaced with γ_{pf2}, l_2, R_2, 0 and $-\varphi_{2B}$ in formulas (13) – (18). Then the DE in the shaft is derived by using (7) (the integration is performed with the help of the MatLab).

The solution of the DE in the shaft is verified by applying an approach based on a direct integration in the time domain [5]. Thus, u_{01} is expressed as

$$u_{01} = \int_0^t \left(\tau_{\eta 11}\dot{\gamma}_{sh11} + \tau_{\eta 12}\dot{\gamma}_{sh12}\right) dt, \tag{22}$$

where the derivatives, $\dot{\gamma}_{sh11}$ and $\dot{\gamma}_{sh12}$, are found by differentiating the expressions for γ_{sh11} and γ_{sh12} in (13) and (15) with respect time. The specific strain energy for shaft portion, Q_2Q_3, is derived by replacing of τ_{sh11}, $\dot{\gamma}_{sh11}$, τ_{sh12} and $\dot{\gamma}_{sh12}$ with τ_{sh21}, $\dot{\gamma}_{sh21}$, τ_{sh22} and $\dot{\gamma}_{sh22}$ in (22). Here, τ_{sh21}, $\dot{\gamma}_{sh21}$, τ_{sh22} and $\dot{\gamma}_{sh22}$ are the stresses and the first derivatives of the strains in the two dashpots for shaft portion, Q_2Q_3. The DE in the shaft is determined by integrating u_{01} and u_{02} in the portions of the shaft. The DE obtained by the two approaches are identical which verifies the solution.

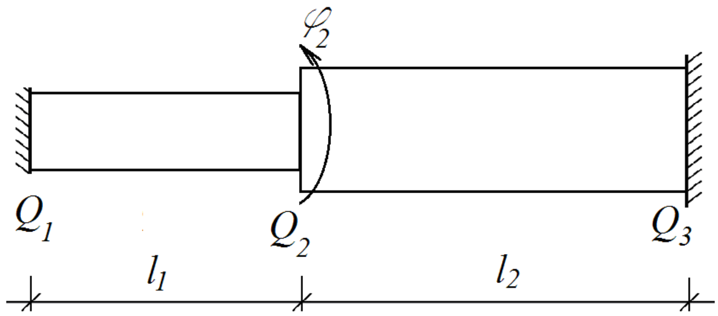

Fig. 5. Statically indeterminate stepped shaft with two portions.

The DE is determined also for the statically indeterminate shaft under angle of twist, φ_2, shown in Figure 5. The shaft sizes and the viscoelastic model are the same as these of the statically determinate shaft in Figure 3. By using (8), (9), (10) and (11) we derive

$$\gamma_{pf1} = -\varphi_2 \frac{R_1}{l_1}, \quad \gamma_{pf2} = \varphi_2 \frac{R_2}{l_2}. \tag{23}$$

After that the DE in the shaft in Figure 5 is determined by substituting of $\varphi_{1B} = 0$ in the first formula in (17) and than applying formulas (13), (14), (15) and (21). Formula (21) is used also to obtain the specific strain energy, u_{02}, in the two dashpots for the shaft portion, Q_2Q_3. For this purpose, γ_{pf1}, l_1, R_1, φ_{1B} and φ_{2B} are replaced with γ_{pf2}, l_2, R_2, 0 and $-\varphi_{2B}$ in formulas (13) – (18).

Verification is carried-out by using (22) and performing replacements as described for the statically determinate shaft.

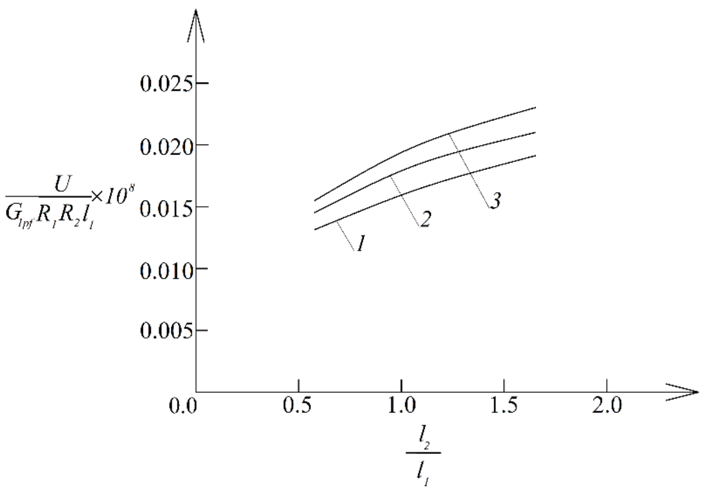

Fig. 6. DE as a function of l_2 / l_1 ratio (curve 1 – at $G_{1cr} / G_{1pf} = 0.5$, curve 2 – at $G_{1cr} / G_{1pf} = 1.0$ and curve 3 – at $G_{1cr} / G = 2.0$).

Numerical results are derived and displayed in Figure 6, Figure 7 and Figure 8.

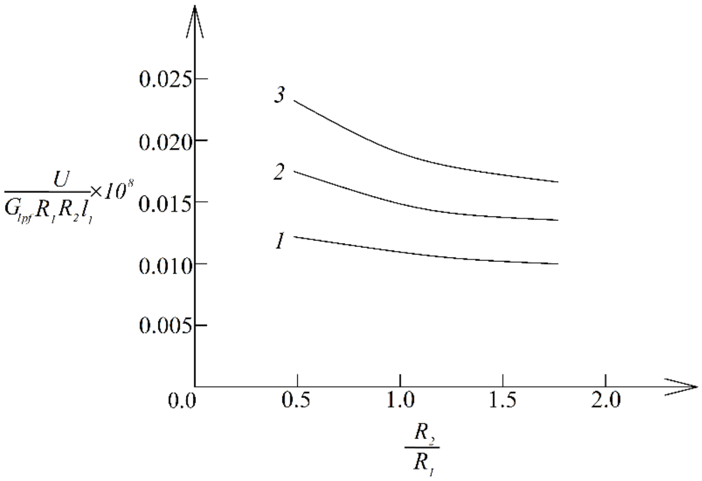

Fig. 7. DE as a function of R_2 / R_1 ratio (curve 1 – at $\eta_{1cr} / \eta_{1pf} = 0.5$, curve 2 – at $\eta_{1cr} / \eta_{1pf} = 1.0$ and curve 3 – at $\eta_{1cr} / \eta_{1pf} = 2.0$).

The following data are used: $l_1 = 0.200$ m, $l_2 = 0.400$ m, $R_1 = 0.005$ m, $R_2 = 0.010$ m, $\varphi_{1B} = 0.0012$ rad, $\varphi_{2B} = 0.0008$ rad, $\omega = 0.0002$, $\mu_1 = 0.7$, $\mu_2 = 0.7$, $\mu_3 = 0.8$ and $\mu_4 = 0.8$.

Figure 6 visualizes the change of the DE in the statically determinate stepped shaft caused by increasing of l_2 / l_1 and G_{1cr} / G_{1pf} ratios. Inspection of the curves in Figure 6 indicates that the DE rises when l_2 / l_1 ratio grows. Similar trend is observed for the influence of G_{1cr} / G_{1pf} ratio, i.e. the DE rises with growth of G_{1cr} / G_{1pf} ratio (Figure 6).

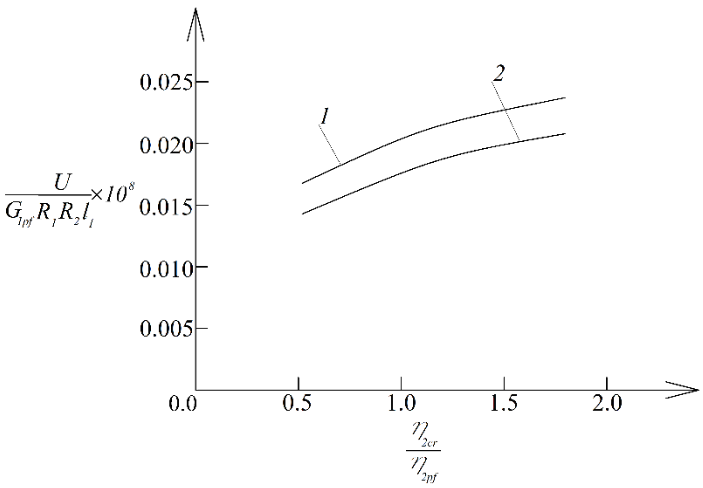

Fig. 8. DE as a function of η_{2cr} / η_{2pf} ratio (curve 1 – in statically determinate shaft and curve 2 – in statically indeterminate shaft).

The trends in the behavior of the DE in the statically determinate stepped shaft at increase of R_2 / R_1 and η_{1cr} / η_{1pf} ratios are illustrated in Figure 7. Analysis of the curves in Figure 7 reveals that the DE reduces as a result of rise of R_2 / R_1 ratio. It can be seen in Figure 7 that rise of η_{1cr} / η_{1pf} ratio leads to growth of the DE.

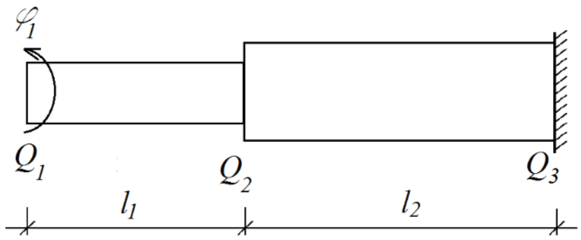

Fig. 9. Stepped shaft under one angle of twist.

The effect of the rise of η_{2cr} / η_{2pf} ratio on the DE in the statically indeterminate shaft is studied. The results are visualized in Figure 8. The DE rises when η_{2cr} / η_{2pf} ratio grows (Figure 8). Similar trend can be observed also for the effect of η_{2cr} / η_{2pf} ratio on the DE in the statically determinate shaft (Figure 8). It can be seen that the DE in the statically indeterminate shaft is lower in comparison with that in the determinate shaft (Figure 8).

It should be specified that the common solution of the DE can be applied also in cases when the shafts are under angles of twist whose number is less than the number of the shaft portions. Such a case is shown in Figure 9. Here, the shaft is under one angle of twist (this is angle, φ_1, varying with time according to (10)). Thus, by applying (1), we have

$$\varphi_1 = \frac{\gamma_{pf1}}{R_1} l_1 + \frac{\gamma_{pf2}}{R_2} l_2 . \tag{24}$$

There are two unknowns, γ_{pf1} and γ_{pf2}, in equation (24). Therefore, we need one complementary equation. Such equation is worked out by considering the equilibrium of the elementary forces on the left and right sides of section, Q_2, of the shaft, i.e.

$$\int_0^{R_1} 2\pi\tau_{Q1Q2} R^2 dR = \int_0^{R_2} 2\pi\tau_{Q2Q3} R^2 dR , \tag{25}$$

where τ_{Q1Q2} and τ_{Q2Q3} are the shear stresses in shaft portions, Q_1Q_2 and Q_2Q_3, respectively. These shear stresses are defined as $\tau_{Q1Q2} = \left(\gamma_{pf1} R\tilde{G}\right) / R_1$ and $\tau_{Q2Q3} = \left(\gamma_{pf2} R\tilde{G}\right) / R_2$ where $\tilde{G}$ is the time-dependent shear modulus of the viscoelastic model. Equations (24) and (25) are solved with respect to γ_{pf1} and γ_{pf2}. Then the DE is derived by (3), (5), (6) and (7). Verification is carried-out via (22).

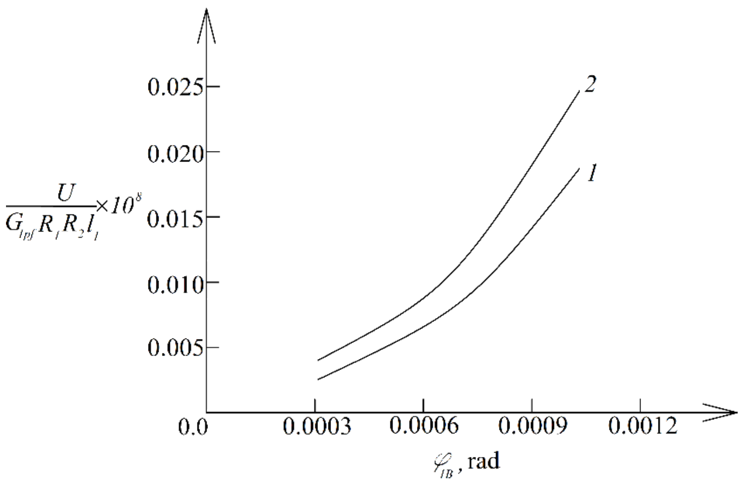

Fig. 10. DE as a function of φ_{1B} (curve 1 – in shaft under one angle of twist and curve 2 – in shaft under two angles of twist).

Variation of the DE with increasing of φ_{1B} in the shaft that is under one angle of twist (Figure 9) is shown in Figure 10. Increase of φ_{1B} induces growth of the DE (Figure 10). The DE is plotted versus φ_{1B} in Figure 10 also for the shaft that is under two angles of twist (Figure 3). It can be seen in Figure 10 that the DE in the shaft under one angle of twist is lower.

It can be generalized that the common solution of the DE is applicable also in the cases when the number of the angles of twist is less than the number of the shaft portions. In such cases, complementary equations have to be worked out by considering equilibrium of characteristic sections of the shaft (the number of the complementary equations is equal to the difference between the number of shaft portions and the number of the angles of twist).

It should be mentioned that the viscoelastic model in Figure 4 and the laws for change of the angles of twist (11) and (12) are used here with purpose to show how the common solution is applied. Certainly, when solving a particular energy dissipation problem, other (appropriate for this particular problem) viscoelastic models and laws for change of the angles of twist can also be used.

4. Conclusion

Common solution of the energy dissipation problem in viscoelastic continuously inhomogeneous stepped shafts under predefined time-dependent angles of twist is derived. Both statically determinate and statically indeterminate shafts are considered. The common solution is applied for analyzing the energy dissipation in a shaft with two portions. The solution is verified by using an approach based on direct integration in time domain. The analysis indicates that the DE rises when l_2 / l_1 and G_{1cr} / G_{1pf} ratios grow. Similar trend in the behavior of the DE is observed at growth of η_{1cr} / η_{1pf} and η_{2cr} / η_{2pf} ratios. Increase of R_2 / R_1 ratio causes a reduction of the DE. It is observed also that the DE in the statically indeterminate shaft is lower compared to that in the statically determinate shaft.

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